MATH-329 Nonlinear optimization
Exercise session 14: Semidefinite programming

Instructor: Nicolas Boumal
TAs: Antoine Gonon, Guifré Sánchez

Document compiled on December 17, 2025

1. Minimum ellipsoid containing points. Given a $n \times n$ positive definite matrix $X \succ 0$, the set

$$\mathcal{E}_X = \{z \in \mathbb{R}^n \mid z^\top X z \leq 1\}$$

is an ellipsoid centered at the origin. Conversely, any (centered) ellipsoid can be described with such a positive definite matrix. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ and consider to problem of finding the ellipsoid of minimal volume, centered at the origin, that contains the points $v_1, \ldots, v_m$. The volume of $\mathcal{E}_X$ is proportional to $(\det X^{-1})^{1/2}$ so minimizing the volume is equivalent to minimizing $\log\det X^{-1} = -\log\det X$. Consequently, the problem can be written as

$$\min_{X \succ 0} -\log\det X \qquad \text{subject to} \qquad v_i^\top X v_i \leq 1 \text{ for all } i \in \{1, \ldots, m\}.$$

This is a convex problem because both the objective function and the constraints are convex. Download the data on Moodle: it contains the points $v_1, \ldots, v_m \in \mathbb{R}^n$ with $m = 10$ and $n = 2$. Use CVX to solve the problem.

2. Maximum cut. Consider a graph with adjacency matrix $A \in \mathrm{Sym}(n)$. Remember that the maximum cut problem is

$$\min_{x \in \mathbb{R}^n} x^\top A x \qquad \text{subject to} \qquad x_i \in \{-1, +1\} \text{ for all } i \in \{1, \ldots, n\}.$$

This is a hard combinatorial problem in general. As shown in the lecture, it is equivalent to

$$\min_{X \succeq 0} \langle A, X \rangle \qquad \text{subject to} \qquad \mathrm{diag}(X) = 1 \text{ and } \mathrm{rank}(X) = 1. \tag{P}$$

If we drop the rank constraint we obtain a semidefinite program.

$$\min_{X \succeq 0} \langle A, X \rangle \qquad \text{subject to} \qquad \mathrm{diag}(X) = 1. \tag{Q}$$

We call it the relaxed problem. There are efficient tools to solve it. Download the data on Moodle: it contains an adjacency matrix A of size $n = 60$.

1. Use CVX to solve the relaxed problem (Q) on MATLAB. The matrix A is in a sparse format. To compute the inner product $\langle A, X \rangle$ you can use the operation `A(:)'*X(:)`.
2. What is the rank of the solution you found? Is it feasible for problem (P)? If not, compute the dominant eigenvector and round it to obtain an approximate solution to (P).